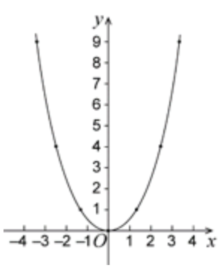
A Torah Basis for Limits and Mathematical Infinity

Miriam Saffern

Like a magnet drawn toward two poles, man oscillates within a dialectic that pervades his entire being. God imbued man with this inner tug-of-war the moment He formed his body out of the dust of the earth. HaRav Joseph B. Soloveitchik ascribes Rashi's dual commentary on the words "וַיִּיצֶר ה אֱלֹקִים אֶת הָאָדָם עָפָר מִן הָאֲדָמָה" (Bereishis 3:6) as the source for this dialectic. First, God created man using the dust of all four corners of the earth, deeming man "cosmic," as the Rav puts it. With the cosmic mindset, man embarks on a quest to discover the endlessness of the world in three ways: through his pursuit of vast knowledge, through his emotional desire to experience all of the world, and through his movement away from home and adaptation to new surroundings [1]. To behold only this perception of his reality, however, would be contradictory to a Jewish existence. Thinking that he can grasp endlessness and vastness would be a fatal flaw, because man is finite and earthly: he has limits.

In mathematics, there is a parallel concept called a limit. A limit is a description of the behavior of a graph, which shows that f(x) becomes arbitrarily close to a number L as x approaches a number c. This is denoted by lim f(x) = L [2]. For example, when the function $f(x) = (x^3-2x^2)/(x-2)$ is graphed on x and y axes, it looks like this:



The two arrows at the edges of the parabola indicate that the graph continues on indefinitely. Despite that, if we were to say that x approaches 2 from the right and the left, the limit would be 4. This can be found numerically by constructing a table of values of f(x) as x becomes arbitrarily closer to 2.

x approaches 2 from the left → ← x approaches 2 from the right

x	1.99	1.999	1.999	2	2.0001	2.001	2.01
f(x)	3.9601	3.996001	3.996001	?	4.00040001	4.0401	4.0401

As x approaches 2, f(x) approaches 4.

In Rashi's second explanation of the pasuk in Bereishis 3:6, he shows that God collected the dirt from one specific spot, the site of the future Beis Hamikdash, and molded it to create "origin-minded man." In this respect, man remains connected to his origin and eventually returns to it. Though opposite in nature, both the origin-minded and cosmic consciousness compel man to seek God. When man, majestic and cosmic, searches for endlessness, it is there that he finds God in the grandeur and farness of His infinite nature. In contrast, origin-minded man meets God in the closeness and finiteness of a single spot [1].

This description of God being both majestas Dei and humilitis Dei begs the question: how could Hashem be both finite and infinite [1, 6]? Before we answer this question, let us describe the infinite nature of God as expressed in Torah.

In the tefillah of Adon Olam, written by Kabbalist Rav Shlomo ibn Gabirol, Hashem is praised as the Master of the Universe, One Who is "timeless, infinite, and omnipotent." Specifically, the words b'li raishis b'li tachlis, that Hashem is without a beginning and without an end, express the fact that He is infinite and does not conform to any laws of nature. A similar concept is packaged into the fourth of the thirteen principles of emunah, which acknowledges that Hashem is the first to exist and the last to exist. In other words, Hashem is the only One that can be infinite [3].

Though these statements clearly show that Hashem is infinite, the concept of infinity itself is still difficult to grasp. A pasuk in Yeshaya divides the concept into three understandable parts. In Yeshaya 6:3, Hashem is referred to as "kadosh kadosh kadosh Hashem tzvakos." At first glance, the repetition of the word kadosh seems extraneous. However, based on Rabbi Avraham Sutton's interpretation of the Targum, the words in fact reflect the infinite nature of God as being threefold: He transcends space, He transcends time, and He is so great and exalted that He is even beyond the comprehension of the heavenly angels [4]. A further discussion of mathematical infinity can elucidate these three concepts of Hashem's infinite nature.

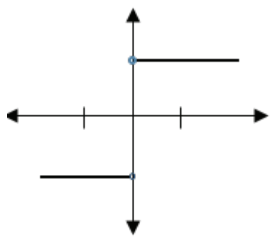
In the study of limits, there are three limits that fail to exist, two of which are pertinent to our discussion. The first is a function which, as x approaches a specific value, approaches one point from the left and a different point from the right. The function (abs(x))/x is an example of this. As x approaches 0 from the right, the function nears 1, but as x approaches 0 from the left, the function nears -1. Plug in arbitrary values for x that approach 0 from each direction, and it becomes clear that the limit differs from the left and the right [2].

x	4	3	2	1	.5	.1	.01	.001	.0001
f(x)	1	1	1	1	1	1	1	1	1

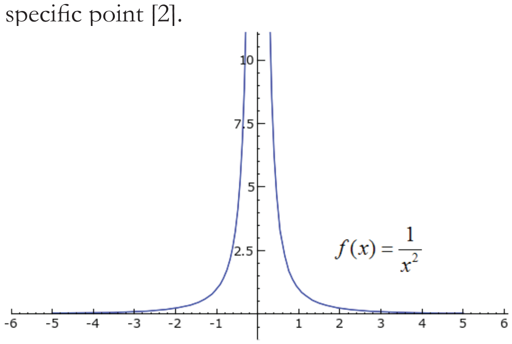
x	-4	-3	-2	-1	-.5	-.1	-.01	-.001	-.0001
f(x)	-1	-1	-1	-1	-1	-1	-1	-1	-1

As depicted in the table, all positive values of x give the function a value of 1, and all negative values of x give the function a value of -1. Since there is no one distinct value that defines the function as x approaches 0, the function has no limit as x approaches 0 [2].

The graph of the function also shows that as x approaches 0, y approaches both 1 and -1, which proves that the limit of (abs(x))/x as x approaches 0 does not exist.



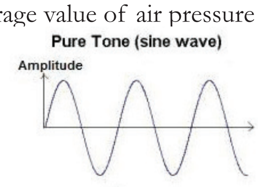
The second type of limit that fails to exist is one for which f(x) does not approach a specific number L as x approaches a given number c. In other words, the function increases or decreases indefinitely, so there will be a point c for which the function does not have a limit. The graph below displays such behavior with the function $f(x) = 1/x^2$. When x approaches 0, f(x) continues to increase and does not reach a specific point [2].



This introduces the concept of infinite limits. An infinite limit is a limit in which f(x) becomes infinitely large or small as x approaches c. Using this definition, the limit of $1/x^2$ as x approaches 0 can be written as $\lim 1/x^2 = \infty$, where ∞ symbolizes infinity [2].

Both limits described above do not exist because for neither of the functions does the f(x) approach a specific number L as x approaches c. The fact that Hashem transcends space helps draw a distinction between the two types of limits described above. Just like Hashem is not confined to one concrete spot, so too the limit of the first function as x approaches 0, if it existed, would be in two places at once. In the natural world, it is impossible for an object to be in two places at the same time (see graph of (abs(x))/x, whose limit fails to exist). But Hashem, on the other hand, Who is not bound by space, has the ability to be in two places at once. He therefore has no end, which is comparable to the limit of the function $f(x) = 1/x^2$ as x approaches 0. There, the function will approach the y axis but will never actually reach it (see previous graph).

The study of music can explain the second concept of Hashem's infinitude, His transcendence of time. A special type of sound wave, called a pure tone, can be represented by a single sine wave. A sine wave is a function of time in which the curves rise and fall between some average value of air pressure [5].



According to French philosopher Francois Marie Charles Fourier, the sum of several pure tones, or sine waves, denotes a sound. The sine waves that comprise the sound are whole number multiples of the lowest frequency wave. A component of a sound wave with a specific frequency f0 is said to be the fundamental of that wave. Each successive component of the wave, with frequency nf0 where n is an integer greater than one, is referred to as the nth harmonic. To accurately portray a sound wave using Fourier's theory, it is necessary to add an infinite number of harmonics. However, mathematicians use a finite sum to create a close depiction of the wave. In theory, a Fourier integral, which adds the harmonics that comprise the sound wave, can accurately represent a sound wave. Yet, in truth, a musical sound does not last forever—it eventually ceases. Therefore, using sine waves, which continue on indefinitely, to represent a finite sound would violate mathematical theory [5].

Mathematicians explain this pitfall by saying that simply the idea of sine waves, not the accurate representation of them, is used to analyze sound [5]. I would venture to say that this mathematical artifice can be explained by the fact that Hashem is the only One Who transcends time. Since sound is natural, and is therefore bound by time, the Fourier theory is not entirely flawless.

Now that the two main concepts describing the nature of the infiniteness of Hashem have been established, let us attempt to use mathematical theory to elucidate the third concept of His infinitude by reconciling a statement in the Gemara that opens an even more baffling question regarding the infiniteness of Hashem. In Daniel 7:10, Daniel recalls a dream in which he witnessed the angels hovering beside Hashem's throne and serving Him. At that moment, there were one million celestial beings serving Him, and altogether ten thousand times ten thousand under His power. A seemingly contradictory observation is documented in Iyov 25:3: "Is there any number of His armies?" These words reflect the infinitude of Hashem's army, unlike the phrase in Daniel, which records an exact number. The Gemara in Chagiga 13b resolves the contradiction by noting that the pasuk in Daniel refers to the number of angels within one troop, and the pasuk in Iyov refers more generally to Hashem's troops, which are endless and have no number. However, we are still left with an enigma. Clearly Hashem is greater than everything and is far superior to his servants, so how can his troops be infinite if He is infinite? Put in other terms, is it possible that there is more than one level of infinity [6]?

In mathematical terms, infinity, denoted by ∞, is a quantity that is greater than any finite quantity. A set of numbers is infinite when there is no last number. For example, the set of all positive integers is infinite, because it consists of the numbers 1, 2, 3, 4, and so on and so forth, indefinitely. Georg Cantor, who pioneered the study of mathematical infinity, called this set ℵ. A set that contains all the squares of integers, {1, 4, 9, 16…}, seems to have fewer elements than ℵ, as it does not include integers which are not perfect squares, such as 2, 3, 5, and 6. However, that would only be true if the two sets were analyzed by matching a number in one set with the identical number in the other set. 1 in ℵ would be paired with 1 in the set of squares of integers, and similar pairs of identical numbers would follow for the numbers 4, 9, 16, and so on, and the non-square integers 2, 3, 5, 6, and so on, would not have a match in the set ℵ. On the other hand, if each integer in ℵ was paired with its corresponding square in the set of squares of integers, the result would be a one-to-one ratio between the elements in $\aleph$ and the elements in the set of the squares of integers. The table below shows such a correspondence, which was discovered by Galileo Galilei [6].

1	2	3	4	5	6	...	N	...
1	4	9	16	25	36	...	n^2	...

Based on this idea of one-to-one correspondence, a countably infinite set can be defined as any set whose elements can be paired with the elements in $\aleph$ and will result in a one-to-one ratio. Such a set is said to have the same number of elements as is in $\aleph$, denoted by $\aleph$. The set of squares of integers, as described above, is therefore countably infinite set with $\aleph$ elements [7, 8]. Conceptually, countable infinity can be compared to the number of stars in the sky [7]. Though the number of the stars is infinite, each star can be assigned a number (see Rashi Shemos 1:1) from the set $\aleph$, resulting in a one-to-one correspondence with the elements in $\aleph$. Therefore, there are $\aleph$ stars.

In addition to countable infinity, there is another type of infinity, called uncountable infinity. Any set that is not countably infinite is said to be uncountably infinite. Such a set is the set of all real numbers, which is greater than mathematical infinity and thus does not conform to the rules of mathematical infinity. One rule of mathematical infinity dictates that for any set X, there is another set, P(X), or the power set, which consists of all the subsets of X. For example, if there is a set X = {1, 2, 3}, which contains three elements, there are 2^3, or eight, possible subsets: {1}, {2}, {3}, {1, 2}, {1, 3}, {2, 3}, the set itself, and the empty set. The power set therefore has a greater number of elements than the original set because if N is the number of elements in X, there exists 2^N elements in P(X). Put simply, for every set $\aleph_0$, there exists a greater set. This does not hold true for uncountable infinity. Since an uncountably infinite set consists of everything that exists in mathematical theory (i.e. all real numbers), it is impossible for any greater set to exist. Similarly, Hashem consists of everything that exists, so there is nothing greater than Him [6].

This answers our above question in the affirmative; there are indeed two levels of infinity. Countable infinity is signified by the Heavenly angels, and uncountable infinity is represented by Hashem. The third aspect of Hashem's infinite nature, that He is so great that He is even beyond the intellectual grasp of the angels, as described above in Yeshaya 6:3, is also illuminated. We see through uncountable infinity that Hashem is the paradigm of greatness: that He is the set of all sets, that He is essentially all that exists. Finally, the concept of uncountable infinity answers our original question regarding man's communion with God in both finiteness and infinitude (i.e. how can Hashem be both finite and infinite?). Just like the uncountably infinite set consists of all finite sets, Hashem, Who is the ultimate infinity, is also the totality of everything natural and finite.

In Derech Hashem, the Ramchal relays that fundamental to our belief in the existence of God is the knowledge that we can never fully understand His true nature. Hashem is incomparable to anything physical, and so we cannot even come close to understanding His nature [9]. This is summed up in the third principle of emunah, in which we proclaim, "I believe with perfect faith that the Creator—may His name be blessed—is not physical, and cannot be perceived by physical means, and there is nothing at all to which He can be compared" [10]. The whole notion of Hashem is shrouded by abstract concepts which are beyond the grasp of human intellect. If a person thinks he can comprehend Hashem's infinite nature, he is once again falling prey to the warped perception that he maintained as a solely cosmic man: that he can understand boundlessness and vastness. What makes the concept of God so difficult to comprehend is the fact that there are no physical terms through which to describe Him. Even mathematical limits and infinity cannot fully describe Him. Rather, the fact that Hashem is infinite shows that mathematical concepts can be derived from Torah. Though we can never comprehend what Hashem is, we will always know that He did exist, does exist, and will exist, everywhere [10].

Acknowledgements:

First and foremost, I would like to express my gratitude to my parents for their infinite support in all my endeavors. More specifically, I would like to thank my mother for providing me with the topic for this manuscript, and my father for helping me find the infinity within Torah and for the time he spent researching ideas. Thank you to all my mathematics teachers throughout my education who helped cultivate my love for mathematics. Thank you to Dr. Loewy for encouraging me to write in this journal, to Dr. S. Weiss for exposing me to the depth and profoundness of Rav Soloveitchik's writings and for directing me to his allusion to the infiniteness of Hashem, and to Dr. Babich for his enthusiasm and interest in my topic.

References

[1] Soloveitchik, Rav Joseph B. Tradition. Majesty and Humility. Jstor.org (retrieved September 9, 2014).
[2] Larson, Ron, Hostetler, Robert P., and Edwards, Bruce H. (2002). Calculus, Seventh Edition. Houghton Mifflin Company, Boston, MA.
[3] The Artscroll Mesorah Series (1984). Siddur Ahavas Shalom, The Complete Artscroll Siddur. Mesorah Publications, Ltd., Brooklyn, NY.
[4] Sutton, Avraham. Temple Institute. The Paradox of Hashem's Transcendence. http://www.templeinstitute.org/107-The-Paradox-of-Transcendence.pdf (retrieved January 19, 2015).
[5] Pierce, John R. (1983). The Science of Musical Sound. Scientific American Books, New York, NY.
[6] Saks, Dr. Tsvi Victor (1999). Different Levels of Infinity in Torah and Mathematics. In B'Or Ha'Torah. pp 113-120.
[7] Kitak, Kim. Countable Infinity and Uncountable Infinity. http://profstewart.org/pm1/talks_09/infinity.pdf/ (accessed January 19, 2015).
[8] Wolfram Math World. http://mathworld.wolfram.com/CountablyInfinite.html/ (accessed January 22, 2015).
[9] Luzzatto, Rabbi Moshe Chaim, Kaplan, Rabbi Aryeh (1997). The Way of God. Feldheim Publishers, Jerusalem, Israel.
[10] Pincus, Rav Shimshon Dovid (2010). Nefesh Shimshon: Gates of Emunah. Feldheim Publishers, Jerusalem, Israel.